an arbitrary orthogonal curvilinear coordinate system. This analogy is also detected between solutions of axisymmetric problems for a continuous layer (half-space) and a layer (half-space) with an absolutely stiff and smooth cylindrical inclusion.

In fact, the kernels of the integral transforms used in solving the problems mentioned satisfy the equation

$$D^2 f = -\gamma^2 f \tag{5}$$

It follows from (2), (4), (5) that the algorithms to determine the transformant of the invariant functions σ_z, τ, w, F agree in the cases mentioned. Therefore, the most tedious part of the solution can be used in examining a number of problems. The same refers to the solutions of the second kind represented in Cartesian and curvilinear coordinates.

The inversion formulas, expressions for the load transformant and results of a computation are understandably distinct for analogous problems.

The order taken to analyze inhomogeneous bodies is also used in a variant of the finite strip method (interpolation method) proposed in [5, 6] for the consideration of the multilayered bodies, and contributes to shortening the computational operations.

REFERENCES

1. Ter-Mkrtich'ian, L. N., Some problems in the theory of elasticity of inhomogeneous elastic media. PMM Vol. 25, № 6, 1961.
2. Rappoport, R. M., Equilibrium of a half-space comprised of layers which are inhomogeneous in depth (three-dimensional problem). Trudy Leningrad Wood Technology Academy, № 109, 1967.
3. Plevako, V. P., Inhomogeneous layer bonded to a half-space under the action of internal and external forces. PMM Vol. 38, № 5, 1974.
4. Gutman, S. G., General solution of elasticity theory problems in generalized cylindrical coordinates. Izv. B. E. Vedeneev All-Union Scient.-Research Inst. of Hydraulic Engineering, Vol. 37, , 1948.
5. Rappoport, R. M., On the question of constructing solutions of axisymmetric and plane problems of the theory of elasticity of a multilayered medium. Izv. B. E. Vedeneev All-Union Sci.-Res. Inst. Hydraulic Eng'ng., Vol. 73, 1963.
6. Rappoport, R. M., Interpolation solutions of the theory of laminar slab bending. Analysis of Spatial Structures, № 14, Stroiizdat, Moscow, 1971.

Translated by M. D. F.

ON THE PROBLEM OF INTERACTION OF RESONANCES

PMM Vol. 40, № 5, 1976, pp. 959-960

G. G. KHAZINA
(Moscow)

(Received October 16, 1975)

The stability is studied of the neutral equilibrium of a system, in a linear approximation, which has two resonances locked at two frequencies, each of which does not cause instability separately in a second approximation. It is shown that in contrast to the case of independent resonances and those locked at one frequency, stability can be lost (in the same order).

A system of ordinary differential equations with real coefficients

$$dx_\alpha / dt = A_\alpha^\beta x_\beta + A_\alpha^{\beta\gamma} x_\beta x_\gamma + O(|x|^3), \quad \alpha, \beta, \gamma = 1, \ldots, n \tag{1}$$

is considered.

The eigenvalues of the linearized system are assumed pure imaginary and prime. The stability is studied of the equilibrium position $x_d = 0$ with respect to variations in the initial data if the system possesses two third order resonances. The stability is understood according to the Birkhoff definition.

This question has been examined partially in [1]; it has been shown that interaction between two resonances, each of which causes no instability (we call them inessential here, for brevity), will not result in instability if the resonances are independent or locked at one frequency. The case of locking at two frequencies is much more complex than the preceding. Examples of specific systems are presented in [1], which show that stability can be lost because of the interaction between inessential resonances.

The interaction between such resonances is examined in more detail below. This investigation does not encompass all possible variants, but includes an example from [1] as a particular case.

Thus, let the system (1) have the following (inessential) resonances:

$$\lambda_2 - 2\lambda_1 = 0, \quad \lambda_1 + \lambda_2 + \lambda_3 = 0$$

Let us write the normal form of the system being investigated in a second approximation

$$\begin{gathered} y_1^{\cdot} = \lambda_1 y_1 + B_1 y_1^* y_2 + C_1 y_2^* y_3^* \\ y_2^{\cdot} = \lambda_2 y_2 + B_2 y_1^2 + C_2 y_1^*,\ y_3^*,\ \dot{y}_3 = \lambda_3 y_3 + C_3 y_1^* y_2^* \\ y_j^{\cdot} = \lambda_j y_j,\ j = 4, \ldots, l,\ 2l = n \end{gathered} \tag{2}$$

(the equations for the conjugate quantities are analogous). Let us introduce the polar coordinates $y_\alpha = \rho_\alpha e^{i\varphi_\alpha}$, $\alpha = 1, \ldots, l$. Then (2) becomes (equations for φ_α are not written down)

$$\begin{aligned} &d\rho_i^2/dt = 2P_j(\psi_1)\rho_1^2\rho_2 + 2Q_j(\psi_2)\rho_1\rho_2\rho_3, \quad j = 1, 2 \\ &d\rho_3^2/dt = 2Q_3(\psi_2)\rho_1\rho_2\rho_3 \\ &\frac{d\psi_1}{dt} = 2\rho_1^2\rho_2\left(\frac{P_1'}{\rho_1^2} + \frac{P_2'}{2\rho_2^2}\right) + 2\rho_1\rho_2\rho_3\left(-\frac{Q_1'}{\rho_1^2} + \frac{Q_2'}{2\rho_2^2}\right) \\ &\frac{d\psi_2}{dt} = \rho_1^2\rho_2\left(-\frac{P_1'}{\rho_1^2} + \frac{P_2'}{\rho_2^2}\right) + \rho_1\rho_2\rho_3\left(\frac{Q_1'}{\rho_1^2} + \frac{Q_2'}{\rho_2^2} + \frac{Q_3'}{\rho_3^2}\right) \\ &2P_j = \beta_j \cos\psi_1 + \alpha_j \sin\psi_1, \quad j = 1, 2 \\ &2Q_k = \mu_k \cos\psi_2 + \gamma_k \sin\psi_2, \quad k = 1, 2, 3 \\ &\beta_1 = 2\operatorname{Re} B_1, \quad \alpha_1 = -2\operatorname{Im} B_1, \quad \beta_2 = 2\operatorname{Re} B_2, \quad \alpha_2 = 2\operatorname{Im} B_2 \\ &\mu_k = 2\operatorname{Re} C_k, \quad \gamma_k = 2\operatorname{Im} C_k, \quad k = 1, 2, 3, \quad \psi_1 = \varphi_2 - 2\varphi_1, \quad \psi_2 = \varphi_1 + \varphi_2 + \varphi_3 \end{aligned} \tag{3}$$

Let us examine the class of systems (3) for which $\beta_j = \mu_k = 0$, $j = 1, 2$, $k = 1, 2, 3$, so that

$$\begin{aligned} &d\rho_j^2/dt = \alpha_j\rho_1^2\rho_2 \sin\psi_1 + \gamma_j\rho_1\rho_2\rho_3 \sin\psi_2, \quad j = 1, 2 \\ &d\rho_3^2/dt = \gamma_3\rho_1\rho_2\rho_3 \sin\psi_2 \\ &\frac{d\psi_1}{dt} = \rho_1^2\rho_2\left(\frac{\alpha_1}{\rho_1^2} + \frac{\alpha_2}{2\rho_2^2}\right)\cos\psi_1 + \rho_1\rho_2\rho_3\left(-\frac{\gamma_1}{\rho_1^2} + \frac{\gamma_2}{2\rho_2^2}\right)\cos\psi_2 \end{aligned} \tag{4}$$

$$\frac{d\psi_2}{dt} = \rho_1^2\rho_2\left(-\frac{\alpha_1}{2\rho_1^2} + \frac{\alpha_2}{2\rho_2^2}\right)\cos\psi_1 + \rho_1\rho_2\rho_3\left(\frac{\gamma_1}{2\rho_1^2} + \frac{\gamma_2}{2\rho_2^2} + \frac{\gamma_3}{2\rho_3^2}\right)\cos\psi_2$$

The inessentiality condition for the first resonance means that $\alpha_2 = -k\alpha_1,\ k > 0$ [1]. The requirement for inessentiality of the second of the resonances being studied means that among the $\gamma_1, \gamma_2, \gamma_3$ there is at least one change of sign [2]. Let us assume that $\gamma_1\gamma_2\gamma_3 \neq 0$.

The system (4) has the integral

$$I = k\rho_1^2 + \rho_2^2 - \varkappa\rho_3^2, \quad \varkappa = \frac{k\gamma_1 + \gamma_2}{\gamma_3}$$

Hence, if $\varkappa < 0$, then the equilibrium is stable.

Now, let $\varkappa > 0$. Let us investigate under what constraints on the coefficients the system (4) has a growing solution of the invariant ray type

$$\rho_1(t) = b(t),\ \rho_2(t) = k_2 b(t),\ \rho_3(t) = k_3 b(t),\ k_2,\ k_3 > 0 \qquad (5)$$
$$db^2/dt > 0, \quad b(0) > 0, \quad \psi_1 = \psi_2 = \pi/2.$$

As is easy to see, it is necessary to require that γ_3 be positive and k_3 should satisfy the quadratic equation

$$\gamma_1 z^2 + \alpha_1 z - \gamma_3 = 0 \qquad (6)$$

and

$$k_2^2 = k_3(-k\alpha_1 + k_3\gamma_2)/\gamma_3$$

Therefore, a solution of the form (5) of the system (4) exists under the following conditions:

$$\gamma_3 > 0, \quad -k\alpha_1 + k_3\gamma_2 > 0, \quad k\gamma_1 + \gamma_2 > 0 \qquad (7)$$

where k_3 is a positive root of Eq.(6) (it exists for any α_1 if $\gamma_1 > 0$ and for $\alpha_1 > 0$ if $\gamma_1 < 0$).

Thus it has been shown that the equilibrium of the system (4) is stable if $(k\gamma_1 + \gamma_2)/\gamma_3 < 0$, and is unstable upon compliance with the inequalities (7).

It is seen that for

$$\gamma_3 < 0, \quad -k\alpha_1 + k_3\gamma_2 > 0, \quad k\gamma_1 + \gamma_2 < 0$$

the equilibrium is also unstable since a growing solution of the form (5) exists, where $\psi_1 = \psi_2 = -\pi/2$.

Let us note that a domain remains in the space of coefficients of the system (4) in which the instability, if it exists, has a different character. In the case $\gamma_1 > 10,\ \gamma_3 > 0$ we have stability for $\gamma_2 < -k\gamma_1$ and instability for $k\alpha_1/k_3 < \gamma_2 < 0$. For $\gamma_1 < 0,\ \gamma_3 > 0$ (then $\gamma_2 > 0$), we have stability for $0 < \gamma_2 < -k\gamma_1$, and instability for $\gamma_2 > k\alpha_1/k_3$. The interval between stability and instability zones will be the smaller, the smaller the γ_3 and will shrink to a point for the degenerate system in which $\gamma_3 = 0$.

REFERENCES

1. Khazina, G. G., Certain stability questions in the presence of resonances. PMM Vol. 38, № 1, 1974.
2. Kunitsyn, A. L., Stability in the critical case of three pairs of pure imaginary roots in the presence of internal resonance. PMM Vol. 35, № 1, 1971.

Translated by M. D. F.